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Three-dimensional viscous fingering patterns in a lifting Hele-Shaw cell : a simulation study

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Abstract : A hierarchical pattern is formed by viscous fingers in a lifting Hele-Shaw cell. If the defending fluid is visco-plastic, a permanent three-dimensional pattern is formed which is usually approximated as quasi-two-dimensional. The viscous fingers show a hierarchical pattern due to competition between growing fingers in a converging geometry. Using a simple algorithm, a Monte-Carlo simulation is presented which attempts to reproduce the real three-dimensional pattern in rectangular geometry. The finger-length distribution, rate of increase of coverage of the invading fluid, the height profile of the final pattern is calculated and their variation studied with changes in the characteristics of the fluid.

Keywords : Viscous fingering, pattern formation, computer simulation, Hele-Shaw cell.

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1. Introduction

Viscous fingering (VF) patterns [1,2] belong to the class of instabilities which are easily produced experimentally but quite complex to study analytically. Existing work mostly focusses on the single Saffman-Taylor finger in a rectangular Hele-Shaw (HS) cell and details of its stability and shape [3]. The circular pattern in a lifting Hele-Shaw cell, formed by competition and interaction between a large number of fingers is however, a fascinating problem [4,5] but much less studied.

When a less viscous fluid is forced into a more viscous fluid under pressure, the interface between the two becomes unstable and the less viscous 'invading' fluid (fluid 1) enters the more viscous 'defending' fluid (fluid 2) in the form of irregular finger like protrusions which may repeatedly branch and sometimes form a fractal pattern. The Hele-Shaw cell, consisting of two parallal glass plates, with a very small gap in between, is the most popular model for study of VF patterns. The more viscous fluid 2, is kept sandwiched between the two plates, and the less viscous fluid 1 is forced in the gap, through a small hole at the centre of the upper glass plate. Fluid 1 forms complex patterns, depending on the characteristic properties of the two fluids. The viscosities of the fluids, their interface tension, the rate of flow of the invading fluid are some of the properties affecting the pattern. Many modifications of this simple HS cell have been developed.

One such apparatus is the 'lifting Hele- Shaw cell' (LHSC). In this modification of the conventional Hele-Shaw cell, the gap between the two plates increases with time. The upper plate is lifted slowly keeping it parallel to the lower one. This causes a sucking effect drawing in the lower viscosity fluid, which surrounds the high viscosity fluid sandwiched between the plates. Quite a large number of papers have been published on the

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LHSC [4,5,6,7], some theoretical and numerical studies have also been done [8]. An equivalent setup is studied as a 'negative squeeze film', by other groups [9–11] who are interested principally in the cavitation that occurs in lubrication problems. If the defending fluid in the LHSC is a visco-plastic paste, the pattern on the two plates after separation, is a permanent three-dimensional pattern, it is actually the complement of the fingers, where the displaced fluid has accumulated. The ridges between two adjacent fingers show a variation in height, and there are peaks at the forks where two ridges join. The height of the ridges is less at the edges and increases towards the centre of the circular plate. This interesting pattern has been studied experimentally [4,12] but we do not know of any attempt at simulating it.

In the present paper, we report a Monte-Carlo simulation of VF patterns in a Hele-Shaw cell with a visco-plastic material such as oil-paint, displaced by a less viscous fluid which may be air or a newtonian liquid. We study characteristics of the pattern, such as survival exponent of the fingers growing upto different lengths, rate of increase in coverage of the displacing fluid with time, and the time required for breakthrough as well as the morphology of the patterns of both the displacing and the displaced fluid. The displaced fluid being visco-plastic piles up when displaced adding an extra dimension (both metaphorically and physically) to the problem. All these features change with parameters representing viscosity contrast, interface tension and density of the fluids. We compare our results with experiments on a lifting Hele-Shaw cell (LHSC) using different fluids.

2. Mathematical modelling

For the normal HS cell which can be approximated as two-dimensional, Laplace's equation

$$\nabla^2 p = 0 \tag{1}$$

with the appropriate boundary conditions gives the pressure distribution $p(x,y,t)$. Assuming the invading fluid to be non-viscous, a constant high pressure can be assigned to it, then the contour with that pressure gives the finger pattern.

In the LHSC, the vertical z-dimension has to be taken into account. The fluid velocity u is now three-dimensional. The incompressibility condition

$$\nabla . u = 0 \tag{2}$$

implies that the fluid flowing in from the sides, moves upwards as the upper plate is lifted. Following [14], we may write

$$\partial u_x/\partial x + \partial u_y/\partial y + \partial u_z/\partial z = 0. \tag{3}$$

Using the lubrication approximation

$$\partial u_z/\partial z = (1/h)dh/dt, \tag{4}$$

where h is the instantaneous separation between the plates. The lateral components of the fluid velocity can be expressed as

$$u_x = -(h^2/12\mu)(\partial p/\partial x) \tag{5}$$

and

$$u_y = -(h^2/12\mu)(\partial p/\partial y) \tag{6}$$

using Darcy's law [13]. Thamida *et al* [14] assume that h is a function of t and independent of position. In that case, h can be expressed in terms of the area of the pattern to get a two-dimensional equation for the pressure

In the present work however, we wish to simulate the three-dimensional structure of the pattern. The height distribution of the ridges as the visco-plastic fluid accumulates between the fingers, is of interest to us. The visco-plastic fluid or paste, is of course non-Newtonian, but we may assume it to be a Bingham fluid, with a constant coefficient of viscosity μ when the stress exceeds the yield stress. We cannot get a simple mathematical representation of the problem and solve for $h(x,y,t)$. So we resort to the Monte-Carlo simulation described below. As a constraint, we keep the total volume of paint constant, and assume a constant lifting force which causes the rate of separation of the plates to increase with time.

3. The simulation algorithm

Growth of viscous fingers and diffusion limited aggregates (DLA) generated by random walks are both examples of Laplacian growth [1]. So random walk algorithms have been used to simulate viscous fingers. However, this has problems because such models produce highly branched and irregular fingers similar to surface deposition models [15]. Different methods are used for smoothening the structures to account for the effect of interface tension [1]. Another problem is that there is no provision in such models for taking into account the physical properties of the two fluids and observing the changes in morphology when these properties change. In our model, we calculate the pressure gradient in the direction of growth (say y) assuming a linear variation from a constant low pressure isobar, and assign growth probability in proportion to the pressure gradient [eq.6], according to Darcy's law [13]. The proportionality constant involves a parameter

·senting the viscosity contrast between the two fluids. lateral flow in the x-direction, normal to finger .th, we assume the pressure gradient to have a constant ·e [eq. (5)]. We assign another parameter which nds on the visco-plastic nature of the defending :, making the probability of displacement smaller, ·n a large amount of it has piled up at a site. In tion, there is a parameter representing surface tension ·h smooths out indentations in a finger during growth. find that the different morphological features observed in the LHSC experiment, can be reproduced by controlling these three parameters. We further quantify our results by recording the exponents governing power-laws connecting certain features of growth, these are compared with experimental results where possible.

Our pattern is simulated on a rectangular two-dimensional lattice of size ($L_x \times L_y$). We start with an initial uniform 'mass' distribution of the defending fluid (fluid 2) over the whole system, assigning one unit of fluid 2 to each site. This is represented by

$$m(x,y,t_{=0}) = 1$$

for all (x,y). An initial random disturbance is created along one edge at $y = 0$ by introducing a column of fluid 1 with lengths varying randomly between 1 to 5 units at alternate sites. Now, the fluid 1 columns grow both forward in the y direction and sideways in the x-direction, with growth probability determined as follows :

$$P_f = \frac{\nabla p(x,y,t)}{D\mu m(x,y,t)} \qquad (7)$$

and

$$P_s = 1/Dm(x,y,t). \qquad (8)$$

Here, P_f and P_s denote respectively the probabilities for forward and lateral growth. $\nabla p(x,y,t)$ is the pressure gradient at (x,y) at time t, calculated assuming a linear variation from a constant lower pressure at the far end $y = L_y$. μ represents the viscosity contrast between the two media and D, the resistance of fluid 2 to pushing by fluid 1 (this may be related to the yield stress). The growth processes are carried out sequentially, the probability determined using two random number generators.

When fluid 1 proceeds by one site in the forward or lat·al direction, the 'mass' of fluid 2 previously occupying th· site is pushed to the next site. So as the fingers of ·id 1 grow, the fluid 2 accumulates forming ridges w· peaks at the forks. This mimics the visco-plastic na·e of fluid 2 and the increasing separation between th· plates in the LHSC. After completing one sequence of forward and lateral growth, the indentations formed in the fingers are smoothed out with a certain probability determined by the surface tension parameter T. Growth processes are repeated until the breakthrough point, when one of the fingers reaches the other end at $y = L_y$. It may be noted that though we use the term 'breakthrough' as in the normal HS cell, in the LHSC, this is actually the time when the fingers reach the centre and the plates separate from each other.

We record the lengths of all the fingers as they grow, the total coverage by fluid 1 (*i.e.* the number of sites occupied by fluid 1 as function of time) and the time for breakthrough t_{br}. We also record the final distribution of fluid 2, which is three-dimensional since the visco-plastic displaced fluid is allowed to accumulate vertically. So we havea relief map of $m(x,y,t_{br})$. The maximum height h_{max} attained at time t_{br} represents twice the separation of the plates at that time. The factor 2 is due to the fact that the same pattern is formed on the upper plate also. In contrast to the conventional Hele-Shaw cell, the total amount of fluid 2 in the cell, is conserved.

As we want to study the morphology of a particular pattern, each pattern is generated separately, without any averaging. But we generate several (about 10) patterns with the same set of parameters changing the random number sequence in order to get repeated observations of the exponents and other quantities studied, we report average values of these.

4. Results of computer simulations

Some typical fingering patterns generated are shown in Figures 1(a-d). The different fingers are shown in different

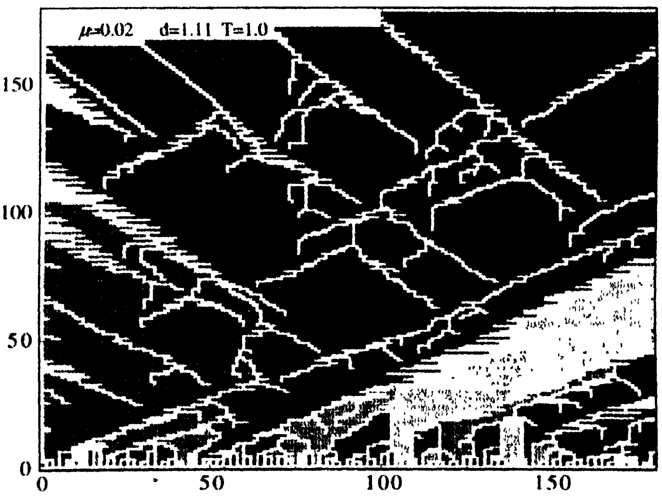

Figure 1a. Finger patterns for high viscosity contrast μ = 0.1 and large nterface tension T = 1. The fingers (of fluid 1) are shown in shade and accumulated fluid 2 in white.

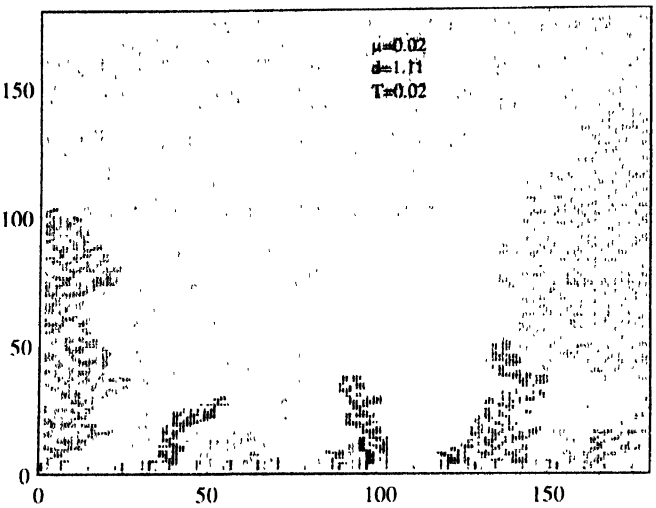

Figure 1b. Finger patterns for high viscosity contrast μ = 0.1 and small interface tension T = 0. The fingers (of fluid 1) are shown in shade and accumulated fluid 2 in white

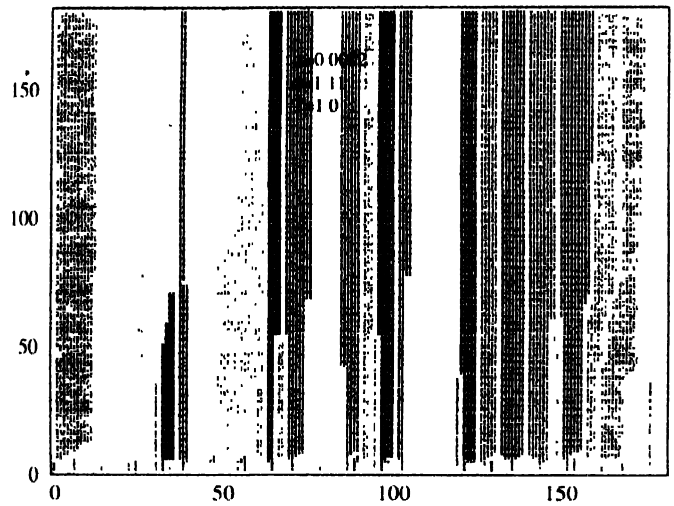

Figure 1c. Finger patterns for low viscosity contrast μ = 0.001 and large interface tension T = 1. The value of D is low here. The fingers (of fluid 1) are shown in shade and accumulated fluid 2 in white.

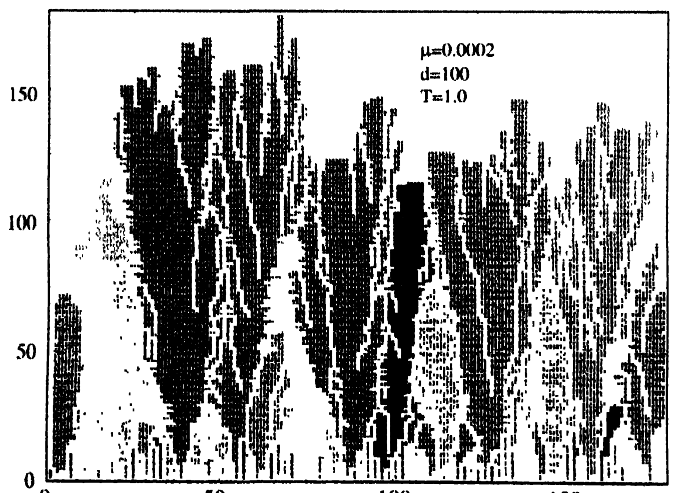

Figure 1d. Finger patterns for low viscosity contrast μ = 0.001 and large interface tension T = 1. The value of D is high here. The fingers (of fluid 1) are shown in shade and accumulated fluid 2 in white.

shades for clarity. The hierarchical character of the patt formed by termination of smaller fingers due to viscous instability, is quite clearly seen. There is a str similarity to the inverted deterministic binary tree. Thi observed in many other natural systems, for exan river basin boundaries [16], and crystal growth in r cavities [17].

There is a noticable change in morphology on vary g the parameters μ, D and T. Decreasing μ decreases he viscosity contrast and enhances forward growth compar d to lateral growth. This makes longer and narrower fingers which grow parallely with much less competition and hence, a slower reduction in number of surviving fingers.

T, the surface tension parameter controls the branching of the fingers. When it is high (here the highest possible value is T = 1), fingers have very little branching and look wide and compact, see Figure 1(a). Low (close to 0) T makes the fingers highly ramified as shown in Figure 1(b).

Table 1 gives the numerical values characterising the pattern morphology for different values of the parameters. We calculate the survival exponent f_{surv}, the breakthrough time t_{br} and the maximum height reached h_{max}.

Table 1. Variation of characteristics of the VF patterns with fluid properties.

μ	T	D	f_{surv}	t_{br}/L_x	h_{max}/L_x
0.1	1.0	1.0	-1.466	1.785	0.51
0.1	1.0	20	-1.380	34.56	0.96
0.1	0.0	1.0	-1.086	1.93	0.16
0.02	1.0	1.0	-1.20	1.04	0.77
0.02	1.0	5.0	-1.09	2.78	0.56
0.02	0.0	5.0	-1.05	3.42	0.14
0.002	1.0	1.0	-0.652	1.0	1.0
0.002	1.0	5.0	-0.733	1.077	0.66
0.002	0.0	1.0	-0.576	1.0	0.49
0.0002	1.0	5.0	-0.484	1.0	1.0
0.0002	0.5	5.0	-0.502	1.0	1.0
0.0002	0.0	5.0	-0.499	1.0	0.66

The highly ramified fingers (in Figure 1(b)) do n t allow the fluid 2 to pile up much and the heig t distribution is more or less uniform. This can be se n from the values of h_{max} in Table 1, h_{max} is quite sm l (for T = 0), except when μ is very low. For very low the straight and parallal fingers reach the end in t shortest time possible on the lattice, *i.e.* in L_y units of time, and fluid 2 is also pushed straight along to its maximum possible height for T = 1. For T = 0, h_{max} is somewhat lower. This can be seen from Figures 1(a-d) and 2(a,b).

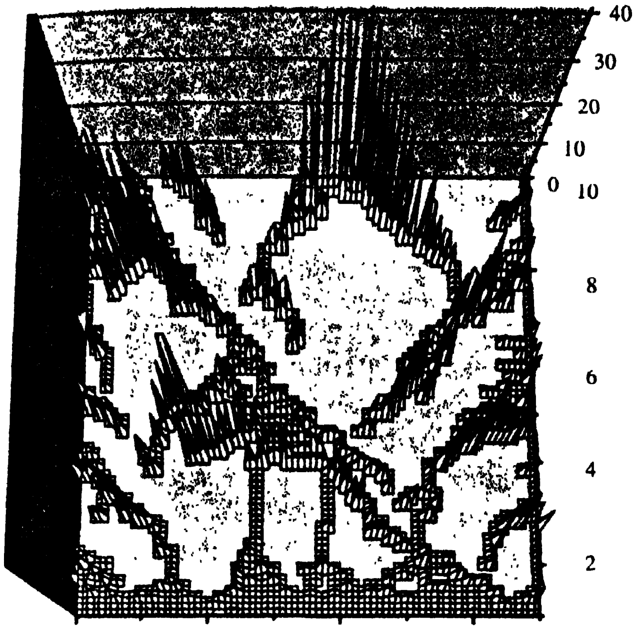

Figure 2a. Three-dimensional representation of the fingering pattern showing accumulation of fluid 2 in ridges, for T = 0. The height distribution of fluid 2 is more or less uniform.

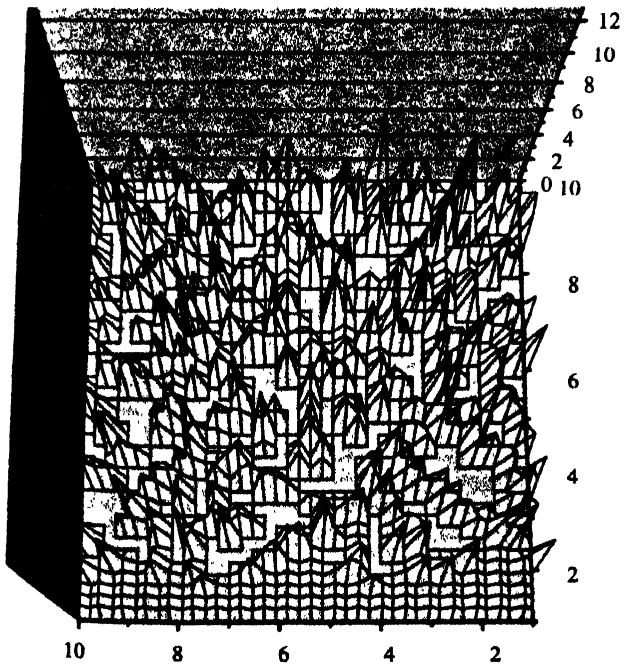

Figure 2b. Three-dimensional representation of the fingering pattern showing accumlation of fluid 2 in ridges, for T = 1. The height distribution of fluid 2 shows strong variation and fingers of fluid 1 are wide and compact.

The effect of D is as follows :

For large D, it is more difficult for fluid 1 to displace fluid 2, after some accumulation, this causes more branching of the fluid 1 fingers and a more uniform distribution of fluid 2. If D is small, there are higher ridges and peaks of fluid 1 towards the far end of the pattern, while fluid 2 fingers look more compact and rounded. These features are exhibited in experimental patterns as we discuss in a later section.

To quantify the overall characteristics of our patterns, we note the following. The number of fingers surviving upto a certain length as the fingers grow. The cumulative length distribution shows a linear behaviour in a log-log plot. Log $N(l)$ *vs* log (l) plot is shown in Figure 3 where

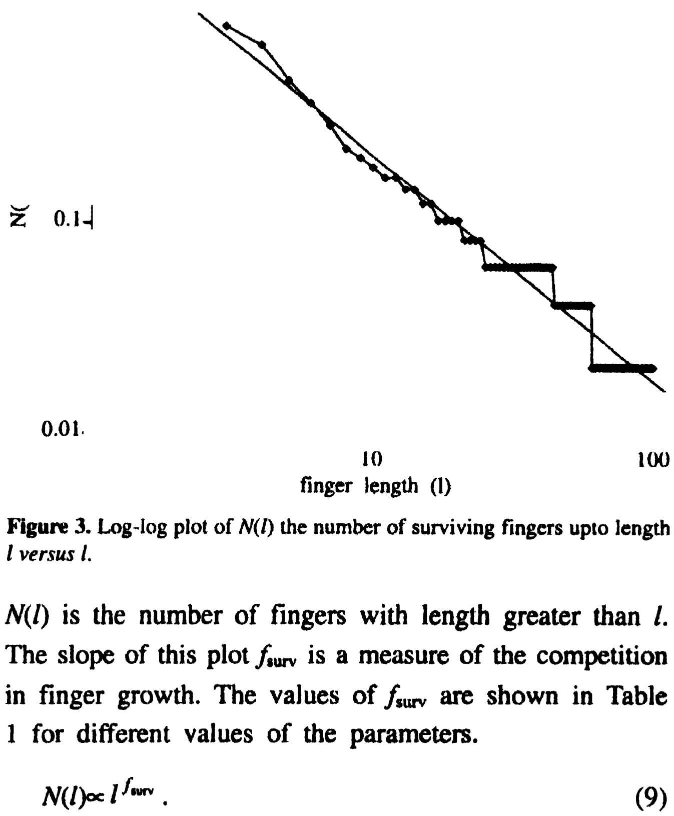

Figure 3. Log-log plot of $N(l)$ the number of surviving fingers upto length l *versus* l.

$N(l)$ is the number of fingers with length greater than l. The slope of this plot f_{surv} is a measure of the competition in finger growth. The values of f_{surv} are shown in Table 1 for different values of the parameters.

$$N(l) \propto l^{f_{surv}} . \tag{9}$$

The graph shows a stepped behaviour for higher values of l, this is connected with the discrete scale invariance observed in LHSC patterns [5]. The stepped region does not always follow the linear fit for log $N(l)$ *versus* log (l) very well, in such cases we report the slope of the initial linear region.

Another feature we study is the increase in total air coverage with time. The coverage $C(t)$, defined as the total area occupied by fluid 1 at time t grows linearly with t upto some time but shows an upward trend later, this is shown in Figure 4. However, a log-log plot does

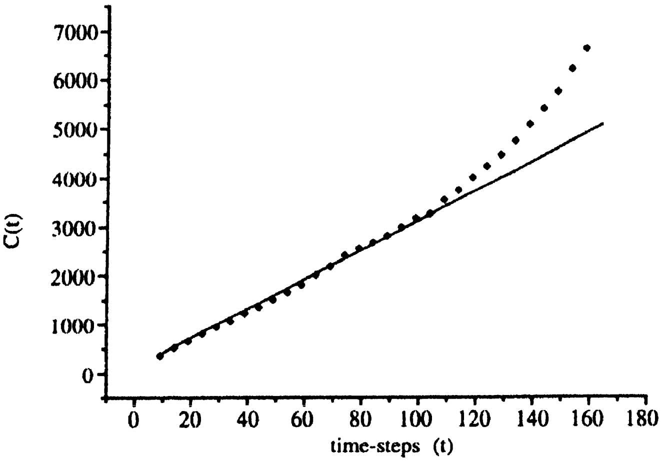

Figure 4. The variation in coverage of fluid 1 *versus* time.

not show linear behaviour for the whole time upto breakthrough t_{br}. The values of t_{br} for different parameters are shown in Table 1 in units of L_y. Also shown is h_{max}, the maximum height upto which the fluid 2 accumulates at breakthrough, this is also in units of L_y.

5. Comparison with experiments on LHSC

We have conducted experiments on the LHSC using different fluid combinations. The experimental setup is described elsewhere in detail [4,5], we give a brief description here. Our lifting Hele-Shaw cell consists of two thick (~0.5 cm) glass plates. The lower one is fixed to a rigid frame, and the upper one can be lifted using a pneumatic cylinder arrangement. The lifting force can be adjusted. In the standard LHSC for studying adhesion, the velocity of plate separation is constant, whereas in our apparatus, the lifting force is controlled and kept constant during the experiment, allowing the velocity to vary. The process of pattern formation is recorded by a CCD camera and analysed using image-pro plus software.

The defending fluid 2 is placed on the lower plate, and the upper plate pressed down on it. This makes the fluid 2 form a circular blob of 2–3 cm diameter. Then the upper plate is slowly lifted, allowing air to enter from the sides, while recording the pattern formation from below the lower plate. If the invading fluid is not air, the appropriate fluid is placed surrounding the defending fluid, taking care that no air bubbles are left in the gap. Formation of the final pattern with the two plates no longer touching, takes typically 20–50 seconds. An identical replica of the pattern is formed on the upper plate as well, due to the symmetry of the arrangement, as the effect of gravity is negligible here. The height of the highest peak on the pattern is therefore, half of the separation between the plates at breakthrough.

In the experiments, we find the behaviour of the fingering pattern is qualitatively similar and the range of the characteristic exponents from experiment and computer simulation match quite well. The survival of fingers shows a power law behaviour on the average with a linear log-log plot on which oscillations or steps are superposed, [see Figures 5(a-b)]. For the high viscosity contrast case (Figure 5a) where fluid 1 is air and fluid 2 is oil-paint,

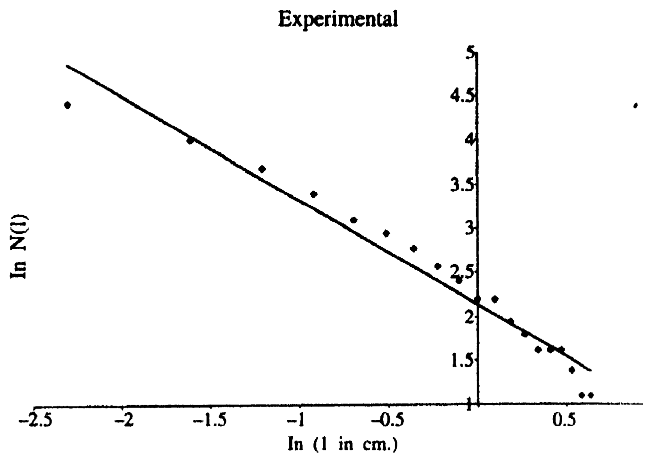

Figure 5a. Experimental results for number of surviving fingers log $N(l)$ vs log l, for high viscosity contrast with air as fluid 1 and oil-paint as fluid 2. Slope of the linear fit f_{surv}(exp) is –0.99.

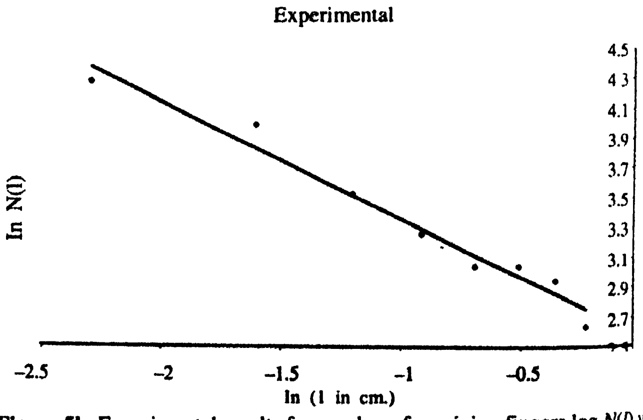

Figure 5b. Experimental results for number of surviving fingers log $N(l)$ vs log l, for low viscosity contrast with water as fluid 1 and oil-paint as fluid 2. Slope of the linear fit f_{surv}(exp) is –0.77.

the slope of the log-log plot of $N(l)$ *versus* l is close to 1, whereas for water-paint combination (Figure 5b) where viscosity contrast is lower (water-paint combination), the slope is less than 1. Appearance of the patterns is also well reproduced. Further details of this experiment is reported elsewhere [18].

As we have used a rectangular lattice for simplicity, we cannot expect the pattern generated to match experiments completely. In the last stages of pattern formation close to breakthrough, the converging circular geometry in the experiment will lead to results very different from what would be produced in a linear cell. But for a blob of fluid 2 with an initial radius large compared to the finger width, initially the pattern of growth will be simiar to the linear case. Of course, linear geometry is not possible in a LHSC experiment with parallel plate separation.

6. Conclusion

The present work is an attempt to generate by computer simulation, the hierarchical pattern of viscous fingers observed in the lifting Hele-Shaw cell. Different characteristics and their variation with properties of the two fluids is presented; the three dimensional nature of the pattern in a visco-plastic defending fluid is taken into account. Predominance of power law behaviour in finger length distribution is observed and the survival exponent f_{surv} takes values ranging from ~0.5 to ~1.5 for increasing viscosity contrast. Variation in height of the pattern is controlled by another parameter D characterising the visco-plastic behaviour of the defending fluid. D also controls the ramification to some extent, large D leads to some branching. However, branching is principally determined by the surface tension parameter T. Oscillations about the power law are indicative of discrete scale invariance [5,19] we propose to investigate this aspect in detail later. Our simulation results are compared with experiments on LHSC patterns generated in our laboratory and we find qualitative as well as quantitative agreement.

We have not yet systematically done an experimental investigation of LHSC patterns, varying the interface tension or yield stress of the fluids, but further work is in progress.

In the present work, we have assumed Darcy's law to be valid, not taking into account the non-Newtonian nature of visco-plastic paste like materials, neither do we consider explicitly the effect of changing plate separation. This should introduce a time-dependent parameter in Darcy's law. We hope to develop a more realistic algorithm in future as the preliminary results are encouraging.

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